CLIFFORD CAUCHY TYPE INTEGRALS ON AHLFORS-DAVID REGULAR SURFACES IN $\mathbb{R}^{m+1}$

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Abstract. The main goal of this paper is centred around the study of the behavior of the Cauchy type integral and its corresponding singular version, both over non-smooth domains in Euclidean space. This approach is based on a recently developed quaternionic Cauchy integrals theory [1, 5, 7] within the three-dimensional setting. The present work involves the extension of fundamental results of the already cited references showing that the Clifford singular integral operator has a proper invariant subspace of generalized Hölder continuous functions defined in a surface of the $(m+1)$-dimensional Euclidean space.

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1. Introduction

Clifford analysis is a generalization of the one-dimensional complex functions theory to higher dimensions. This theory studies the properties of functions with values in Clifford algebras constructed over Euclidean spaces. Among the texts dealing with this thematic we refer the reader to [9, 24, 28, 29, 30, 31, 36, 46, 48, 52] and the references therein. Intense interest in the Clifford analysis is evidenced by the hundreds of reported papers from MathSciNet, Mathematical Reviews on the Web, during the last five years. Some introductory papers to the basic rudiments of the Clifford analysis, which include historical notes have been worked out in [15, 22, 23, 54, 60, 62, 63].
In different latitudes, there have been developed scientific conferences to approach the theme of Clifford analysis and related topics, which gained a lot of attention and meant as a favorable place for the fruitful debate and the clarification of important ideas in this field. In [32], appendix 1, the editors took care to collect information on such events from 1981 to 1999. The state of the art of the Clifford analysis and their applications can be found in [61], see also [8].
Clifford algebras were introduced in the 19-th century by mathematicians and physicists in various attempts to provide a good foundation to geometric calculus in Euclidean space. The discovery of the quaternions by the Irish mathematician W. R. Hamilton in 1843, represented a decisive step in that direction. The geometric algebras are described by W. K. Clifford in the paper [12] (see also [13]). At the present time a number of important geometric ideas underlying Clifford algebras have arisen in several domains of mathematics and mathematical physics. Over the years the construction of Clifford algebras has appeared in the literature; early progress in this direction was made by Chevalley [11].
The power and the elegance of the Analytic Functions Theory in several areas of mathematics lead to the search of similar theories in higher dimensions. Clifford analysis represents one of the more appropriate ways of extending the Analytic Functions Theory to higher dimensions, having the Monogenic Functions Theory, as natural analogue. The structural similarity among both theories and the successful applications to an important number of problems in physics have motivated an intense development of Clifford analysis in the last two decades. A nice illustration of this is the attempt to embrace both Dirac operators and Clifford algebras pointed out by J. Cnops in his recently published book [14].
The subject of Clifford analysis has been discovered and independently redis-

covered in several times in the passed century. One of the first authors who deal with such subject was A. C. Dixon [25], later C. Lanczos described the rudiments of Quaternionic analysis in his doctoral thesis [37] (see also [63]). Among the years 1930 and 1950 the Swiss mathematician Rudolph Fueter [27] and his students published about fifteen papers on the subject. At the same time as Fueter's work appeared, the Rumanian mathematicians Moisil and Theodorescu [47] worked on closely results. But was not until period in the late of 1960 and early of 1970 that the Clifford analysis was considered like a new and independent theory. In this stage we must mention independent papers by Richard Delanghe, David Hestenes and Viorel Iftimie (see [21, 33, 34]). Each of these papers illustrated how many aspects of one-dimensional complex analysis have been extended to higher dimensions using Clifford algebras.

In our days there are several mathematicians and physicists, occupied in the topics of Clifford analysis, that consider Richard Delanghe the main pillar of this field. In Prague (2000) took place the conference "*Clifford Analysis and its Applications*", dedicated to him on the occasion of his 60th birthday. Following H. Malonek, it was not only his mathematics that made Richard Delanghe one of the founder of Clifford analysis, but also for his passion for new ideas, his humanity and for his enthusiastic support to young researchers (see [41]).

The authors would like to dedicate this paper to Richard Delanghe on the occasion of his visit to the University of Holguín and Oriente University in October of 2002. We also gratefully acknowledge him for his enthusiastic support to the promotion and development of the Clifford analysis in Cuba.

Several classical results of Analytic Functions Theory have been extended in a natural way to the Monogenic Functions Theory setting, such as: Cauchy-Riemann's equations, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, and Sokhotski-Plemelj's formulae. Some of these results are treated in this paper, both for its own sake and for its connection with classical harmonic analysis and boundary value problem in minimally smooth domains.

According to Michael Shapiro [58] the principal facts, which conform the base of the Complex analysis are:

- The excellent structure of the complex numbers.
- The factorization of the two-dimensional Laplace operator by the Cauchy-Riemann operator.
- The existence of a Green's formula or two-dimensional of Stokes.

These facts, in a similar way, are present in the Clifford analysis framework, which justifies from the authors point of view that it is considered to be an appropriate variant of generalization of the Analytic Functions Theory in higher dimensions.

In this notes we are going to deal with the Cauchy type integrals associated to the Clifford-Cauchy kernel. It is the intention to give an essentially modern-day approach to the Hölder continuous boundedness estimates for the Cauchy type integral and its corresponding singular version in domains with boundary complicated geometrically involving methods on Clifford analysis. This work provides that the quaternionic results obtained in [1, 5, 7] have been extended nearly.

The study of the paper is part of our efforts for extending some aspects on boundary value problem of Riemann-Hilbert type in non-smooth domain to the Clifford algebra-valued function context. Directly, the obtained results here have come to be a suitable higher dimensional analogues of those previously developed in [1, 6] for quaternionic case. A predecessor of the above intention was developed in [2].

A review report by D. Peña and J. Bory regarding Riemann boundary value problem for monogenic Clifford algebra-valued function in non-smooth domain using singular integral equations will appear in [49].

2. Preliminaries

Let $\mathbb{R}^{m+1}$, the $(m+1)$-dimensional Euclidean space equipped with the norm

$$|x| := (x_0^2 + \ldots + x_m^2)^{1/2}, \text{ for } x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}.$$

An open and connected set $\Omega \subset \mathbb{R}^{m+1}$ will be called domain. We denote by $\mathcal{H}^m$, the m-dimensional Hausdorff measure over $F \subset \mathbb{R}^{m+1}$, which is defined as

$$\mathcal{H}^m(F) := \lim_{\delta \to 0} \inf \Big\{ \sum_{j=1}^{\infty} \frac{\alpha_m}{2^m} (\text{diam } F_j)^m : F \subset \cup_j F_j, \text{ diam } F_j < \delta \Big\},$$

where α_m represents the volume of unitary ball in $\mathbb{R}^m$ (see [38, 53]).

In this paper, method of integrating over the boundaries of the domains will be taking regarding the Hausdorff measure $\mathcal{H}^m$, which represents a natural generalization of the "surface area" measure on domains with sufficiently smooth boundaries.

A good subclass of m-sets (i.e. sets having finite m-dimensional Hausdorff measure) of $\mathbb{R}^{m+1}$ which includes all of the surfaces traditionally considered by most of the authors interested in Clifford Analysis is the subclass of rectifiable sets of H. Federer [26]. It is worth to remember that each of them has outward

pointing normal vector defined $\mathcal{H}^m$-almost everywhere.
Throughout the paper Γ denotes a compact topological surface with diameter d verifying the geometric condition $\mathcal{H}^m(\Gamma) < +\infty$ (to emphasize this we write Γ be an m-surface), which represents a natural condition without any quantitative estimates on the size of the surface Γ.
We consider the functions $\theta_z^m(r) := \mathcal{H}^m(\Gamma_r(z))$ and $\theta^m(r) := \sup_{z\in\Gamma} \theta_z^m(r)$, where $\Gamma_r(z) := \Gamma \cap B_r^m(z)$, and $B_r^m(z)$ denotes the ball centered at z with radius r. Notice that θ_z^m and θ^m are bounded and non-decreasing functions in $(0, d]$, satisfying $\theta_z^m(r) \leq \theta^m(r)$. As far as we know, the functions θ_z^1 and θ^1 were introduced by V. V. Salaev in [55].
If Γ is a rectifiable Jordan curve in the complex plane (which is characterized as those satisfying $\mathcal{H}^1(\Gamma) < +\infty$), it can be parametrized nicely by a Lipschitz function. For m-dimensional surfaces ($m \geq 2$) one can not, in general, find such a nice parametrization. Recall that a surface can have finite m-dimensional Hausdorff measure without being at all a rectifiable surface.
Let $\Omega^+ \subset \mathbb{R}^{m+1}$ be a bounded, oriented and simply connected domain with boundary Γ. By Ω^- we denote the complement domain of $\Omega^+ \cup \Gamma$.
The symbol z_x, represents a point of Γ satisfying $|x - z_x| = \mathrm{d}ist(x, \Gamma)$, where x is an arbitrary element of the space $\mathbb{R}^{m+1}$.

Definition 2.1 *The surface Γ is called Ahlfors-David regular (in short AD-regular) if there exists a positive constant c such that*

$$c^{-1}r^m \leq \theta_z^m(r) \leq cr^m,$$

for all $z \in \Gamma$ and $r \in (0, d]$.

Here and in the whole paper c denotes a positive constant not necessarily the same in different occurrences.
The requirement that the surface Γ be an AD-regular can be viewed as a quantitative version of the property of having upper and lower densities with respect to $\mathcal{H}^m$, which are positive and finite. However, the regularity condition does not imply the existence of the density in any point of the surface Γ. According to a well-known as well beautiful fact a rectifiable set can be viewed as a set which $\mathcal{H}^m$-almost all its points have density as a positive finite number [50]. More information about the AD-regular sets can be found in [17, 18, 19, 20, 42, 56, 57].
In the course of the paper will be of particular importance the Ahlfors-David regularity condition for the surfaces. The particular cases of smooth, Liapunov or Lipschitz surfaces are examples of such regular surfaces, but so are many

countable unions of surfaces and even non rectifiable sets as well as many Cantor type sets.

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$. Consider the 2^m-dimensional real Clifford algebra $\mathbb{R}_{0,m}$ obtained from the generating relations $e_ie_j + e_je_i = -2\delta_{ij}$, $i, j = 1, \ldots, m$. The basic elements $e_A := e_{i_1} \ldots e_{i_k}$ for $A := \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$, with $i_1 < i_2 < \ldots < i_k$, define a basis of $\mathbb{R}_{0,m}$, $e_\emptyset = e_0 = 1$ is the identity of $\mathbb{R}_{0,m}$.

Thus, an arbitrary element $a \in \mathbb{R}_{0,m}$, may be represented as

$$a = \sum_A a_A e_A, \; a_A \in \mathbb{R}$$

and its norm is defined by $|a| := (\sum_A a_A^2)^{1/2}$. Especially the elements $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$ will be identified with

$$\sum_{i=0}^{m} x_i e_i \in \mathbb{R}_{0,m}.$$

It is possible to introduce the Clifford conjugation of $a \in \mathbb{R}_{0,m}$, which is defined as $\overline{a} := \sum_A a_A \overline{e}_A$, where $\overline{e}_A := (-1)^k e_{i_k} \ldots e_{i_2} e_{i_1}$. For the generating elements we have $\overline{e}_i := -e_i$ and for the identity $\overline{e}_0 := e_0$.

The functions $u : \Omega^+ \subset \mathbb{R}^{m+1} \to \mathbb{R}_{0,m}$, may be written as

$$u(x) = \sum_A u_A(x) e_A,$$

where u_A are real valued functions. We say u belongs to a particular function space if every component u_A belongs to the same space.

On this way, for any suitable $F \subset \mathbb{R}^{m+1}$, we say that $u \in \mathcal{C}^p(F, \mathbb{R}_{0,m})$ if each component u_A of u possesses p-th continuous partial derivatives in F. For $p = 0$, $\mathcal{C}^0(F, \mathbb{R}_{0,m})$ represents the space of all Clifford algebra-valued continuous functions, which will be denoted simply by $\mathcal{C}(F, \mathbb{R}_{0,m})$. If u is a bounded continuous function on F, the norm in $\mathcal{C}(F, \mathbb{R}_{0,m})$ will be given by

$$\|u\|_\infty := \sup_{x \in F} |u(x)|.$$

We now define the Cauchy-Riemann operator in $\mathbb{R}^{m+1}$

$$D_m := \sum_{i=0}^{m} e_i (\partial / \partial x_i).$$

Its fundamental solution is given by

$$E_m(x) := \frac{1}{\sigma_m} \frac{\overline{x}}{|x|^{m+1}}, \quad x \in \mathbb{R}^{m+1} \setminus \{0\},$$

σ_m is the area of the unit sphere in $\mathbb{R}^{m+1}$, i.e.

$$D_m E_m = E_m D_m = \delta(y),$$

where $\delta(y)$ stands for the classical δ-function in $\mathbb{R}^{m+1}$.
An $\mathbb{R}_{0m}$-valued function u in the class $\mathcal{C}^1(\Omega^+, \mathbb{R}_{0m})$ is called (left) monogenic in Ω^+ if $D_m u = 0$ in Ω^+. The set of monogenic functions in Ω^+ will be denoted by $M(\Omega^+, \mathbb{R}_{0,m})$.
Considering the conjugate operator

$$\overline{D}_m := \sum_{i=0}^{m} \overline{e}_i \frac{\partial}{\partial x_i},$$

it is easy to show that $D_m \overline{D}_m = \overline{D}_m D_m = \Delta_{m+1}$, where Δ_{m+1} is the Laplace operator in $\mathbb{R}^{m+1}$.
Let us take a look at the definition of the Clifford-Cauchy kernel, in the framework of distribution. We assume that all necessary knowledge on generalized solution in distribution sense is available (see e.g. [65]).

Definition 2.2 *The Clifford-Cauchy kernel in x is the generalized solution of D_m w.r.t. δ_x, δ_x being the point evaluation, i.e., for each $\psi \in \mathcal{D}(\mathbb{R}^{m+1}, \mathbb{R}_{0,m})$*

$$\big\langle \delta_x(y), \psi(y) \big\rangle = \psi(x).$$

Remark 2.1 The Clifford-Cauchy kernel is given by

$$e_x(y) := \frac{1}{\sigma_m} \frac{\overline{y-x}}{|y-x|^{m+1}}, \quad y \neq x.$$

Remark 2.2 As we already know, the generalized solution of D_m w.r.t. the classical δ-function in $\mathbb{R}^{m+1}$ is the fundamental solution of D_m. By the definition itself, we thus have that

$$E_m(y-x) = e_x(y).$$

The definition of the Clifford-Cauchy kernel as we set it here is inspired by the idea of the Cauchy kernel related to the Dirac operator on a manifold as pointed out by J. Cnops in [14].

In Clifford analysis the classical Stokes formula, referred to as boundary theorem (see [3]), tell us that

$$\int_{\Gamma} v(y)n(y)u(y)d\mathcal{H}^m(y) = \int_{\Omega^+}((vD_m)u + v(D_m u))d\mathcal{L}^{m+1}(y)$$

($\mathcal{L}^{m+1}$ denotes the Lebesgue measure in $\mathbb{R}^{m+1}$), where the outward pointing normal (unit) vector $n(y)$ is understood as a Clifford algebra-valued function and the corresponding integrand is interpreted in the sense of the Clifford product. In the usual way the boundary theorem yields the Borel-Pompeiu integral representation formula

$$\int_{\Gamma} E_m(y-x)n(y)u(y)d\mathcal{H}^m(y) - \int_{\Omega^+} E_m(y-x)D_m u(y)d\mathcal{L}^{m+1}(y) =$$

$$= \begin{cases} u(x), & x \in \Omega^+ \\ 0, & x \in \Omega^-, \end{cases}$$

for all function $u \in \mathcal{C}^1(\Omega^+, \mathbb{R}_{0,m}) \cap \mathcal{C}(\Omega^+ \cup \Gamma, \mathbb{R}_{0,m})$.
For the particular case of a smooth surface Γ, these formulae were proved in [9], but it is not so obvious that the Stokes formula remains valid even if the boundary is very complicated geometrically. Research on the problem of finding the most general form of the Stokes formula has contributed greatly to the development of Geometric Measure Theory. The very general notion of exterior normal vector introduced by Federer in [26] involves only the measure theoretic behavior of the boundary with respect to the $(m+1)$-Lebesgue measure $\mathcal{L}^{m+1}$ and imposes no a priori topological restriction on it. This flexible notion permits to establish the validity of the Stokes formula for every open subset Ω of $\mathbb{R}^{m+1}$ whose boundary Γ be an m-surface. In this case $n(y)$ denotes the outward pointing normal (unit) vector to Γ at the point y defined by Federer.
Next we record some auxiliary facts concerning integral estimates that will be relevant for us in the sequel.

Lemma 2.1 *Suppose f be a non-negative and non-increasing function in $(0,d]$. Then, whenever $r', r'' \in (0,d]$, $r'' > r'$, the following equality is valid*

$$\int_{\Gamma_{r''}(z)\setminus\Gamma_{r'}(z)} f(|y-z|)d\mathcal{H}^m(y) = \int_{r'}^{r''} f(\tau)d\theta_z^m(\tau),\ z \in \Gamma.$$

The proof of this lemma is straightforward. One has only to estimate the integral on the left hand-side of the equality.
Because the proofs of the following couple of lemmas resemble the original way developed in [1, 5, 7] for quaternionic case, they are omitted.

Lemma 2.2 *If $z_1, z_2 \in \Gamma$, $|z_1 - z_2| = 2r$, $r \in (0, d/2]$, then*

$$\left|\int_{\Gamma\setminus\{\Gamma_r(z_1)\cup\Gamma_r(z_2)\}} E_m(y-z_1)n(y)d\mathcal{H}^m(y)\right| \leq c.$$

Lemma 2.3 *For $z \in \Gamma$, $r \in (0, d]$, are valid the following estimates.*

i) If $U \in M(\Omega^+, \mathbb{R}_{0,m}) \cap \mathcal{C}(\Omega^+ \cup \Gamma, \mathbb{R}_{0,m})$, then

$$\left|\int_{\Gamma\setminus\Gamma_r(z)} E_m(y-z)n(y)(U(y)-U(z))d\mathcal{H}^m(y)\right| \leq$$
$$\leq c \max_{x\in\Omega^+\cup\Gamma,\ |x-z|=r} |U(x)-U(z)|.$$

ii) If $U \in M(\Omega^-, \mathbb{R}_{0,m}) \cap \mathcal{C}(\Omega^- \cup \Gamma, \mathbb{R}_{0,m})$, $U(\infty) = 0$, then

$$\left|\int_{\Gamma\setminus\Gamma_r(z)} E_m(y-z)n(y)(U(y)-U(z))d\mathcal{H}^m(y) + U(z)\right| \leq$$
$$\leq c \max_{x\in\Omega^-\cup\Gamma,\ |x-z|=r} |U(x)-U(z)|.$$

3. Singular Cauchy Integral Operator

Let $u \in \mathcal{C}(\Gamma, \mathbb{R}_{0,m})$, we consider the singular Cauchy integral operator defined as

$$\mathbf{S}_\Gamma u(z) := 2\int_\Gamma E_m(y-z)n(y)(u(y)-u(z))d\mathcal{H}^m(y) + u(z),\ z \in \Gamma,$$

where the integral which defines the operator $\mathbf{S}_\Gamma$ has to be taken in the sense of Cauchy's principal value, and the function u is such that the following integrals

$$\int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(u(y)-u(z))d\mathcal{H}^m(y)$$

converge uniformly to zero in $z \in \Gamma$, as $\epsilon \to 0$.
The space of all continuous functions on Γ which satisfy the above condition will be denoted by $\mathcal{S}(\Gamma, \mathbb{R}_{0,m})$. Notice that for any continuous function u belonging to $\mathcal{S}(\Gamma, \mathbb{R}_{0,m})$ the singular Cauchy integral $\mathbf{S}_\Gamma u(z)$ exists for any z and it defines a continuous function on Γ.
Since

$$2\int_\Gamma E_m(y-z)n(y)d\mathcal{H}^m(y) = 1,\ z \in \Gamma,$$

when Γ be a smooth surface, it follows that in this case the singular Cauchy integral operator $\mathbf{S}_\Gamma$ coincides with the operator

$$u(z) \to 2\int_\Gamma E_m(y-z)n(y)u(y)d\mathcal{H}^m(y),\ z \in \Gamma.$$

Given a positive real function $\varphi : (0,d] \to \mathbb{R}_+$ with $\varphi(0+) = 0$, it will be called a majorant if $\varphi(\tau)$ is non-decreasing and $\varphi(\tau)/\tau$ is non-increasing, for $\tau \in (0,d]$. If in addition, there exists a constant c such that

$$\int_0^\nu \frac{\varphi(\tau)}{\tau}d\tau + \nu\int_\nu^d \frac{\varphi(\tau)}{\tau^2}d\tau \leq c\varphi(\nu),$$

whenever $\nu \in (0,d]$, then we say that φ is a regular majorant.
We denote by $H_\varphi(\Gamma, \mathbb{R}_{0,m})$ the class of all functions of $\mathcal{C}(\Gamma, \mathbb{R}_{0,m})$, satisfying a generalized Hölder of type

$$\omega_u(\tau) := \tau \sup_{t\geq\tau} t^{-1} \sup_{x,y\in\Gamma,|x-y|\leq t} |u(x)-u(y)| \leq c\varphi(\tau),\ \tau \in (0,d],$$

where φ is a majorant (see [35, 45]).
It is easy to prove that $H_\varphi(\Gamma, \mathbb{R}_{0,m})$ equipped with the norm

$$\|u\|_{H_\varphi} := \|u\|_\infty + \sup_{\tau\in(0,d]} \frac{\omega_u(\tau)}{\varphi(\tau)},$$

is a real Banach space.
This section is devoted to a discussion of the Hölder boundedness of the singular Cauchy integral operator $\mathbf{S}_\Gamma$ on AD-regular surfaces. Nevertheless, we want to point out that a function $u \in H_\varphi(\Gamma, \mathbb{R}_{0,m})$, where φ is a regular majorant, belongs to $\mathcal{S}(\Gamma, \mathbb{R}_{0,m})$ by the assumption that Γ be an AD-regular surface.
In [10, 17, 18, 19, 39, 40, 43, 44, 56] the Clifford algebra-valued Cauchy integrals on one and higher dimensional domains was established. Basically, the study includes the L^2-boundedness of the Hilbert transform, which is of the same kind as that of Coifman-McIntosh-Meyer's (see [16]).
The issue of the L^2-boundedness is not central here, and so we shall not bother.

Theorem 3.1 *Let Γ be an AD-regular surface and φ be a regular majorant. Then $\mathbf{S}_\Gamma$ is a bounded operator from $H_\varphi(\Gamma, \mathbb{R}_{0,m})$ into itself, that is there exists a constant c such that*

$$\|\mathbf{S}_\Gamma u\|_{H_\varphi} \leq c\|u\|_{H_\varphi},$$

for any function $u \in H_\varphi(\Gamma, \mathbb{R}_{0,m})$.

Proof. Suppose $u \in H_\varphi(\Gamma, \mathbb{R}_{0,m})$, φ be a regular majorant, and $z_1, z_2 \in \Gamma$, $|z_1 - z_2| = 2t \leq d$.

$$\mathbf{S}_\Gamma u(z_1) - \mathbf{S}_\Gamma u(z_2) = 2\Big(\int_{\Gamma_t(z_1)} E_m(y - z_1)n(y)(u(y) - u(z_1))d\mathcal{H}^m(y) -$$

$$-\int_{\Gamma_t(z_2)} E_m(y - z_2)n(y)(u(y) - u(z_2))d\mathcal{H}^m(y) +$$

$$+\int_{\Gamma\setminus\{\Gamma_t(z_1)\cup\Gamma_t(z_2)\}} (E_m(y - z_1) - E_m(y - z_2))n(y)(u(y) - u(z_1))d\mathcal{H}^m(y) +$$

$$+\int_{\Gamma\setminus\{\Gamma_t(z_1)\cup\Gamma_t(z_2)\}} E_m(y - z_2)n(y)(u(z_2) - u(z_1))d\mathcal{H}^m(y) +$$

$$+\int_{\Gamma_t(z_2)} E_m(y - z_1)n(y)(u(y) - u(z_1))d\mathcal{H}^m(y) -$$

$$-\int_{\Gamma_t(z_1)} E_m(y - z_2)n(y)(u(y) - u(z_2))d\mathcal{H}^m(y)\Big) + (u(z_1) - u(z_2)) =: \sum_{k=1}^{7} J_k.$$

We first estimate the integral J_1 and J_2 simultaneously:

$$|J_k| \leq c\int_{\Gamma_t(z_k)} \frac{|u(y) - u(z_k)|}{|y - z_k|^m} d\mathcal{H}^m(y) \leq c\int_{\Gamma_t(z_k)} \frac{\omega_u(|y - z_k|)}{|y - z_k|^m} d\mathcal{H}^m(y),$$

for $k = 1, 2$.
Making use of the following inequality (see [28], p. 178)

$$|E_m(y - z_1) - E_m(y - z_2)| \leq \frac{|z_1 - z_2|}{\sigma_m} \sum_{j=0}^{m-1} \frac{1}{|y - z_1|^{j+1}|y - z_2|^{m-j}},$$

we get

$$|J_3| \leq c|z_1 - z_2| \sum_{j=0}^{m-1} \int_{\Gamma\setminus\{\Gamma_t(z_1)\cup\Gamma_t(z_2)\}} \frac{\omega_u(|y - z_1|)}{|y - z_1|^{j+1}|y - z_2|^{m-j}} d\mathcal{H}^m(y).$$

If $|y - z_1| \leq |y - z_2|$, then

$$\frac{\omega_u(|y - z_1|)}{|y - z_1|^{j+1}|y - z_2|^{m-j}} \leq \frac{\omega_u(|y - z_1|)}{|y - z_1|^{m+1}}, \quad j = 0, ..., m - 1.$$

If $|y-z_1| \geq |y-z_2|$, then taking into account that the function $\dfrac{\omega_u(\tau)}{\tau^{j+1}}$ is non-increasing in $(0,d]$, we obtain

$$\frac{\omega_u(|y-z_1|)}{|y-z_1|^{j+1}|y-z_2|^{m-j}} \leq \frac{\omega_u(|y-z_2|)}{|y-z_2|^{m+1}}, \quad j=0,...,m-1.$$

Consequently

$$|J_3| \leq c|z_1-z_2|\sum_{k=1}^{2}\int_{\Gamma\setminus\Gamma_t(z_k)} \frac{\omega_u(|y-z_k|)}{|y-z_k|^{m+1}} d\mathcal{H}^m(y).$$

From Lemma 2.2, we have the estimate

$$|J_4| \leq c\,\omega_u(|z_1-z_2|) \leq c\,\omega_u(\frac{|z_1-z_2|}{2}).$$

Now we proceed to estimate the integral J_5. Taking into account that

$$|y-z_1| \leq |y-z_2| + |z_2-z_1| \leq \frac{3}{2}|z_1-z_2|, \quad y \in \Gamma_t(z_2)$$

and that $\Gamma_t(z_2) \subset \Gamma\setminus\Gamma_t(z_1)$, then

$$|J_5| \leq c|z_1-z_2|\int_{\Gamma_t(z_2)} \frac{\omega_u(|y-z_1|)}{|y-z_1|^{m+1}} d\mathcal{H}^m(y) \leq$$

$$\leq c|z_1-z_2|\int_{\Gamma\setminus\Gamma_t(z_1)} \frac{\omega_u(\ verty-z_1|)}{|y-z_1|^{m+1}} d\mathcal{H}^m(y).$$

In a similar way

$$|J_6| \leq c|z_1-z_2|\int_{\Gamma\setminus\Gamma_t(z_2)} \frac{\omega_u(\ verty-z_2|)}{|y-z_2|^{m+1}} d\mathcal{H}^m(y).$$

Finally we obtain the obvious estimate

$$|J_7| \leq \omega_u(|z_1-z_2|) \leq 2\omega_u(\frac{|z_1-z_2|}{2}).$$

Because of the obtained estimates, we get

$$|\mathbf{S}_\Gamma u(z_1) - \mathbf{S}_\Gamma u(z_2)| \leq c\Big(\sum_{k=1}^{2}\big(\int_{\Gamma_t(z_k)} \frac{\omega_u(|y-z_k|)}{|y-z_k|^m} d\mathcal{H}^m(y) +$$

$$+\frac{|z_1-z_2|}{2}\int_{\Gamma\setminus\Gamma_t(z_k)}\frac{\omega_u(|y-z_k|)}{|y-z_k|^{m+1}}d\mathcal{H}^m(y))+\omega_u(\frac{|z_1-z_2|}{2})\Big).$$

From Lemma 2.1 and inequality $\theta^m_{z_k}(\tau)\le\theta^m(\tau)$ for $\tau\in(0,d]$, $k=1,2$, we get clearly the estimate

$$|\mathbf{S}_\Gamma u(z_1)-\mathbf{S}_\Gamma u(z_2)|\le c\Big(\int_0^t\frac{\omega_u(\tau)}{\tau^m}d\theta^m(\tau)+t\int_t^d\frac{\omega_u(\tau)}{\tau^{m+1}}d\theta^m(\tau)+\omega_u(t)\Big).$$

Taking into account that the right side of the previous expression is non-decreasing function of t, we have

$$|\mathbf{S}_\Gamma u(z_1)-\mathbf{S}_\Gamma u(z_2)|\le c\Big(\int_0^{2t}\frac{\omega_u(\tau)}{\tau^m}d\theta^m(\tau)+2t\int_{2t}^d\frac{\omega_u(\tau)}{\tau^{m+1}}d\theta^m(\tau)+\omega_u(2t)\Big).$$

Since Γ is an AD-regular, we obtain

$$\omega_{\mathbf{S}_\Gamma u}(\epsilon)\le c\Big(\int_0^\epsilon\frac{\omega_u(\tau)}{\tau}d\tau+\epsilon\int_\epsilon^d\frac{\omega_u(\tau)}{\tau^2}d\tau+\omega_u(\epsilon)\Big)\le$$

$$\le c\Big(\int_0^\epsilon\frac{\omega_u(\tau)}{\tau}d\tau+\epsilon\int_\epsilon^d\frac{\omega_u(\tau)}{\tau^2}d\tau\Big),\ \epsilon\in(0,d].$$

Making use of the last calculation and of the inequality

$$\omega_u(\epsilon)\le\|u\|_{H_\varphi}\varphi(\epsilon),\ \epsilon\in(0,d],$$

we get

$$\omega_{\mathbf{S}_\Gamma u}(\epsilon)\le c\|u\|_{H_\varphi}\Big(\int_0^\epsilon\frac{\varphi(\tau)}{\tau}d\tau+\epsilon\int_\epsilon^d\frac{\varphi(\tau)}{\tau^2}d\tau\Big)\le$$

$$\le c\|u\|_{H_\varphi}\varphi(\epsilon),\ \epsilon\in(0,d].$$

Therefore

$$\mathbf{S}_\Gamma u\in H_\varphi(\Gamma,\mathbb{R}_{0,m}).$$

On the other side, since

$$|\mathbf{S}_\Gamma u(z)|\le c\|u\|_{H_\varphi}\int_0^d\frac{\varphi(\tau)}{\tau}d\tau+\|u\|_\infty\le$$

$$\le c\|u\|_{H_\varphi},\ z\in\Gamma.$$

Thus

$$\|\mathbf{S}_\Gamma u\|_{H_\varphi}\le c\|u\|_{H_\varphi},$$

which implies that $\mathbf{S}_\Gamma$ is a bounded operator on $H_\varphi(\Gamma,\mathbb{R}_{0,m})$. ■

4. The Cauchy Type Integral

With the use of the Clifford-Cauchy kernel, we will introduce the left handed version of the higher dimensional Cauchy type integral $\mathbf{C}_\Gamma$, formally defined by:

$$\mathbf{C}_\Gamma u(x) := \int_\Gamma E_m(y-x)n(y)u(y)d\mathcal{H}^m(y),\ x \notin \Gamma,$$

where $u \in \mathcal{C}(\Gamma, \mathbb{R}_{0,m})$.
It follows without difficulty that $\mathbf{C}_\Gamma u$ enjoys the very nice property of being a monogenic function in $\mathbb{R}^{m+1} \setminus \Gamma$ vanishing at infinity.
We spend this section investigating the existence of continuous limit values of the Cauchy type integral $\mathbf{C}_\Gamma$ with integrand in $\mathcal{S}(\Gamma, \mathbb{R}_{0,m})$. From a geometrical point of view this setting is as general as possible, while our analytical assumption concerning the integrand is also very general.
It is of interest to note that relaxing of the continuity condition on u under which the functions

$$\mathbf{C}^{\pm}_\Gamma u(x) := \begin{cases} \mathbf{C}_\Gamma u(x), & x \in \Omega^\pm \\ \frac{1}{2}(\mathbf{S}_\Gamma u(x) \pm u(x)), & x \in \Gamma, \end{cases}$$

are continuous in the closed domains $\Omega^\pm \cup \Gamma$ leads us to an improvement of the well-known Sokhotski-Plemelj formulae.
For $u \in \mathcal{C}(\Gamma, \mathbb{R}_{0,m})$ we define the following operators

$$\mathcal{L}^k_\epsilon(u,z,x) := \mathbf{C}_\Gamma u(x) - (2-k)u(z) -$$

$$-\int_{\Gamma \setminus \Gamma_\epsilon(z)} E_m(y-z)n(y)(u(y)-u(z))d\mathcal{H}^m(y),\ k=1,2,$$

$x \in \Omega^\pm$ respectively, $z \in \Gamma$, and $\epsilon \in (0,d]$.

Lemma 4.1 *Suppose Γ be an AD-regular surface, $u \in \mathcal{C}(\Gamma, \mathbb{R}_{0,m})$ and $|x - z_x| = \epsilon$. Then*

$$|\mathcal{L}^k_\epsilon(u,z_x,x)| \le c\Big(\omega_u(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big),\ k=1,2.$$

Proof. It is not difficult to see that for $k = 1, 2$

$$\mathcal{L}_\epsilon^k(u,z_x,x)=\int_{\Gamma_\epsilon(z_x)}E_m(y-x)n(y)(u(y)-u(z_x))d\mathcal{H}^m(y)+$$

$$+\int_{\Gamma\setminus\Gamma_\epsilon(z_x)}(E_m(y-x)-E_m(y-z_x))n(y)(u(y)-u(z_x))d\mathcal{H}^m(y)=:J_1+J_2.$$

Taking into account that $|y-z_x|\leq\epsilon$ for $y\in\Gamma_\epsilon(z_x)$ and the function ω_u is non-decreasing, then

$$|J_1|\leq c\int_{\Gamma_\epsilon(z_x)}\frac{\omega_u(|y-z_x|)}{|y-x|^m}d\mathcal{H}^m(y)\leq c\,\omega_u(\epsilon)\int_{\Gamma_\epsilon(z_x)}\frac{d\mathcal{H}^m(y)}{|y-x|^m}.$$

But as $|y-x|\geq|x-z_x|=\epsilon$ and Γ be an AD-regular surface, this implies

$$|J_1|\leq c\frac{\omega_u(\epsilon)}{\epsilon^m}\theta_{z_x}^m(\epsilon)\leq c\,\omega_u(\epsilon).$$

We turn now our attention to J_2. As $|y-z_x|\leq|y-x|+|x-z_x|\leq 2|y-x|$, then we have

$$|E_m(y-x)-E_m(y-z_x)|\leq c\frac{|x-z_x|}{|y-z_x|^{m+1}},$$

then

$$|J_2|\leq c|x-z_x|\int_{\Gamma\setminus\Gamma_\epsilon(z_x)}\frac{\omega_u(|y-z_x|)}{|y-z_x|^{m+1}}d\mathcal{H}^m(y).$$

According to the Lemma 2.1

$$|J_2|\leq c|x-z_x|\int_\epsilon^d\frac{\omega_u(\tau)}{\tau^{m+1}}d\theta_{z_x}^m(\tau)\leq c|x-z_x|\int_\epsilon^d\frac{\omega_u(\tau)}{\tau^{m+1}}d\theta^m(\tau).$$

Finally the assertion follows from of the regularity condition of Γ and of the estimates of J_1,J_2. ■

Theorem 4.1 *Suppose Γ be an AD-regular surface, $u\in\mathcal{S}(\Gamma,\mathbb{R}_{0,m})$. Then*

$$\mathbf{C}_\Gamma^\pm u\in\mathcal{C}(\Omega^\pm\cup\Gamma,\mathbb{R}_{0,m}).$$

Proof. For the sake of brevity we restrict to the case $\mathbf{C}_\Gamma^+u$. Let z be a fixed point of Γ, $x\in\Omega^+$, set $|x-z_x|=\epsilon$, then we have

$$\mathbf{C}_\Gamma u(x)-\frac{1}{2}(\mathbf{S}_\Gamma u(z)+u(z))=\mathcal{L}_\epsilon^1(u,z_x,x)-$$

$$-\int_{\Gamma_\epsilon(z_x)} E_m(y-z_x)n(y)(u(y)-u(z_x))d\mathcal{H}^m(y)+$$

$$+\frac{1}{2}(\mathbf{S}_\Gamma u(z_x)-\mathbf{S}_\Gamma u(z))+\frac{1}{2}(u(z_x)-u(z)).$$

Hence

$$|\mathbf{C}_\Gamma u(x)-\frac{1}{2}(\mathbf{S}_\Gamma u(z)+u(z))|\leq|\mathcal{L}^1_\epsilon(u,z_x,x)|+$$

$$+\left|\int_{\Gamma_\epsilon(z_x)} E_m(y-z_x)n(y)(u(y)-u(z_x))d\mathcal{H}^m(y)\right|+$$

$$+|\mathbf{S}_\Gamma u(z_x)-\mathbf{S}_\Gamma u(z)|+|u(z_x)-u(z)|.$$

On account of the continuity of u and $\mathbf{S}_\Gamma u$ on Γ, Lemma 4.1 leads readily to the desired result. ■

Remark 4.1 The Sokhotski-Plemelj formulae suggest to introduce also an analogous operator $\mathbf{S}^*_\Gamma$ to the singular Cauchy integral operator. Let Γ be an m-surface and $u\in\mathcal{C}(\Gamma,\mathbb{R}_{0,m})$. If there exists a function Φ monogenic in $\mathbb{R}^{m+1}\setminus\Gamma$, with the additional condition $\Phi(\infty)=0$, whose restrictions Φ_{Ω^+} and Φ_{Ω^-} are continuous into the closure of the corresponding domains $\Omega^\pm$, and the difference of the boundary values $\Phi^\pm$ of these restrictions coincide with u in the whole Γ, then the function Φ is unique and one can set

$$\mathbf{S}^*_\Gamma u:=\Phi^+ + \Phi^-.$$

If Γ is an AD-regular surface, then the operators $\mathbf{S}_\Gamma$ and $\mathbf{S}^*_\Gamma$ coincide in the space $\mathcal{S}(\Gamma,\mathbb{R}_{0,m})$.
Another important remark is that

$$\int_\Gamma E_m(y-z)n(y)(\mathbf{C}^+_\Gamma u(y)-\mathbf{C}^+_\Gamma u(z))d\mathcal{H}^m(y)=0.$$

Thus as a consequence, we infer that for all functions $u\in\mathcal{S}(\Gamma,\mathbb{R}_{0,m})$, Γ being an AD-regular surface, the following identity holds.

$$\int_\Gamma E_m(y-z)n(y)(\mathbf{S}_\Gamma u(y)-\mathbf{S}_\Gamma u(z))d\mathcal{H}^m(y)+$$

$$+\int_\Gamma E_m(y-z)n(y)(u(y)-u(z))d\mathcal{H}^m(y)=0.$$

The above is clearly seen to be equivalent with $\mathbf{S}_\Gamma^2 = \mathcal{I}$, where $\mathcal{I}$ denotes the identity operator, i.e., $\mathbf{S}_\Gamma$ is an involution on $\mathcal{S}(\Gamma, \mathbb{R}_{0,m})$.

Remark 4.2 There is a tradition to show how singular Cauchy integral in Clifford analysis could be used to get information about singular Bochner-Martinelli type integral M_m, $m \geq 2$ in multidimensional Complex analysis (see [59]). Making essential use of the involution property of the singular Cauchy integral, it seems to be natural to consider the problem about a formula for M_m^2. This work has been already done in [51] applying that Bochner-Martinelli type integral is "a part" of the singular Cauchy integral.
The formula studied there restricted on Liapunov surfaces coincides with the one obtained in [64], by the Quaternionic analysis methods, for $m = 2$, but also on the same environment of Liapunov surfaces.

5. Invariant Subspace

The Theorem 3.1 tells us that $H_\varphi(\Gamma, \mathbb{R}_{0,m})$ represents an invariant subspace for the operator $\mathbf{S}_\Gamma$. It is well-known that when an operator or class of operators is shown to have invariant subspaces, a general structure theory usually emerges (see [4]).
In this section what we want to show is that the subspace

$$\mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m}) := \Big\{ u \in H_\varphi(\Gamma, \mathbb{R}_{0,m}) : \sup_{\tau \in (0,d]} \frac{\Theta_u(\tau)}{\varphi(\tau)} < +\infty \Big\},$$

where

$$\Theta_u(\tau) := \tau \sup_{t \geq \tau} t^{-1} \sup_{\epsilon \in (0,t], z \in \Gamma} \left| \int_{\Gamma_\epsilon(z)} E_m(y-z) n(y)(u(y) - u(z)) d\mathcal{H}^m(y) \right|,$$

is also invariant subspace for the singular Cauchy integral operator on AD-regular surfaces.
On this, a norm can be defined

$$\|u\|_{\mathcal{Z}_\varphi} := \|u\|_{H_\varphi} + \sup_{\tau \in (0,d]} \frac{\Theta_u(\tau)}{\varphi(\tau)}.$$

We would like to point out that the case $m = 2$, has been discussed in [7]. In the whole section we will consider Γ being an AD-regular surface.
Let us begin with the following two lemmas.

Lemma 5.1 *If $u \in \mathcal{S}(\Gamma, \mathbb{R}_{0,m})$, then for $z \in \Gamma$*

$$\sup_{x\in\Omega^+\cup\Gamma\ ,|z-x|=\epsilon} |\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma^+ u(x)| \leq c\Big(\omega_u(\epsilon) + \Theta_u(\epsilon) +$$

$$+\epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2}d\tau\Big),\ \epsilon \in (0, d].$$

Proof. Let us see the two possible cases.
Case 1. Suppose that $|z-x| = \epsilon$, $x \in \Omega^+$ and set $|x - z_x| = \nu$. It is not difficult to see that

$$\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma u(x) = -\int_{\Gamma_\nu(z_x)} E_m(y-x)n(y)(u(y)-u(z_x))d\mathcal{H}^m(y) -$$

$$-\int_{\Gamma\setminus\Gamma_\nu(z_x)} (E_m(y-x) - E_m(y-z_x))n(y)(u(y)-u(z_x))d\mathcal{H}^m(y) +$$

$$+\int_{\Gamma_\nu(z_x)} E_m(y-z_x)n(y)(u(y)-u(z_x))d\mathcal{H}^m(y) +$$

$$+\frac{1}{2}(\mathbf{S}_\Gamma u(z) - \mathbf{S}_\Gamma u(z_x)) + \frac{1}{2}(u(z) - u(z_x)) =: \sum_{k=1}^5 J_k.$$

Due to the similar arguments in the proof of Lemma 4.1 and also taking into account that $\nu \leq \epsilon$, we can derive the inequality

$$|J_1| + |J_2| \leq c\Big(\omega_u(\epsilon) + \epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2}d\tau\Big).$$

For J_3 we have

$$|J_3| \leq \Theta_u(\nu) \leq \Theta_u(\epsilon).$$

As $|z - z_x| \leq |z-x| + |x - z_x| \leq 2|z-x| = 2\epsilon$, then

$$|J_4| \leq \omega_{\mathbf{S}_\Gamma u}(\epsilon) \text{ and } |J_5| \leq \omega_u(\epsilon).$$

On the other hand, starting from the reasonings in the proof of Theorem 3.1 and using the definition of the characteristic metric Θ_u, we obtain that

$$|J_4| \leq \omega_{\mathbf{S}_\Gamma u}(\epsilon) \leq c\Big(\omega_u(\epsilon) + \Theta_u(\epsilon) + \epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2}d\tau\Big).$$

Consequently

$$|\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma u(x)| \le c\Big(\omega_u(\epsilon) + \Theta_u(\epsilon) + \epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big),$$

for $|z-x| = \epsilon$, $x \in \Omega^+$.
Case 2. Let $x \in \Gamma$, such that $|z-x| = \epsilon$, then

$$\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma^+ u(x) = \frac{1}{2}(\mathbf{S}_\Gamma u(z) - \mathbf{S}_\Gamma u(x)) + \frac{1}{2}(u(z) - u(x)).$$

For that

$$|\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma^+ u(x)| \le \omega_{\mathbf{S}_\Gamma u}(\epsilon) + \omega_u(\epsilon),$$

hence

$$|\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma^+ u(x)| \le c\Big(\omega_u(\epsilon) + \Theta_u(\epsilon) + \epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big).$$

The statement of the lemma follows now from the above considered cases. ■

Lemma 5.2 *If* $u \in \mathcal{S}(\Gamma, \mathbb{R}_{0,m})$, $\epsilon \in (0,d]$, *then*

$$\Theta_{\mathbf{S}_\Gamma u}(\epsilon) \le c\Big(\omega_u(\epsilon) + \Theta_u(\epsilon) + \epsilon\int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big).$$

Proof. For $z \in \Gamma$ we have $\mathbf{S}_\Gamma u(z) = 2\mathbf{C}_\Gamma^+ u(z) - u(z)$. Therefore

$$\Big|\int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{S}_\Gamma u(y) - \mathbf{S}_\Gamma u(z)) d\mathcal{H}^m(y)\Big| \le \Theta_u(\epsilon) +$$

$$+2\Big|\int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y)\Big|.$$

As we have

$$\int_\Gamma E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y) = 0,$$

for that

$$\Big|\int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y)\Big| =$$

$$= \left| \int_{\Gamma \setminus \Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y) \right|.$$

With this at hand, from Lemma 2.3 we obtain that

$$\left| \int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y) \right| \leq$$

$$\leq c \sup_{x \in \Omega^+ \cup \Gamma\ , |z-x|=\epsilon} |\mathbf{C}_\Gamma^+ u(z) - \mathbf{C}_\Gamma^+ u(x)|.$$

Therefore from Lemma 5.1

$$\left| \int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{C}_\Gamma^+ u(y) - \mathbf{C}_\Gamma^+ u(z)) d\mathcal{H}^m(y) \right| \leq c\Big(\omega_u(\epsilon) +$$

$$+\Theta_u(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big).$$

For that

$$\left| \int_{\Gamma_\epsilon(z)} E_m(y-z)n(y)(\mathbf{S}_\Gamma u(y) - \mathbf{S}_\Gamma u(z)) d\mathcal{H}^m(y) \right| \leq c\Big(\omega_u(\epsilon) +$$

$$+\Theta_u(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_u(\tau)}{\tau^2} d\tau\Big),$$

this completes the proof. ■

Before proceeding to the proof of the main result of the section, let us note some connection between the spaces $H_\varphi(\Gamma, \mathbb{R}_{0,m})$ and $\mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m})$.

Proposition 5.1 *Let φ be a majorant such that $\int_0^d \frac{\varphi(\tau)}{\tau} d\tau < +\infty$. Then φ_1 defined by $\varphi_1(\nu) := \int_0^\nu \frac{\varphi(\tau)}{tau} d\tau$, $\nu \in (0, d]$, is a majorant and $H_\varphi(\Gamma, \mathbb{R}_{0,m}) \subset Z_{\varphi_1}(\Gamma, \mathbb{R}_{0,m})$.*

This is essentially the same as one proved in [7] for the quaternionic-valued function spaces.

Remark 5.1 Note that if φ is also a regular majorant, then $H_\varphi(\Gamma, \mathbb{R}_{0,m}) = \mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m})$.

Slight modifications of the reasoning in [7] as well as the applications of the previous results of the section give the following main theorem.

Theorem 5.1 *Let Γ be an AD-regular surface and φ be a majorant such that*

$$\epsilon \int_{\epsilon}^{d} \frac{\varphi(\tau)}{\tau^2} d\tau \leq c\varphi(\epsilon), \epsilon \in (0, d].$$

Then the operator $\mathbf{S}_\Gamma$ is bounded on the subspace $\mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m})$.

Proof. If $u \in \mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m})$ then for $\epsilon \in (0, d]$ we have

$$\omega_u(\epsilon) \leq \|u\|_{\mathcal{Z}_\varphi} \varphi(\epsilon) \text{ and } \Theta_u(\epsilon) \leq \|u\|_{\mathcal{Z}_\varphi} \varphi(\epsilon),$$

and so

$$\epsilon \int_{\epsilon}^{d} \frac{\omega_u(\tau)}{\tau^2} d\tau \leq \epsilon \|u\|_{\mathcal{Z}_\varphi} \int_{\epsilon}^{d} \frac{\varphi(\tau)}{\tau^2} d\tau \leq c\|u\|_{\mathcal{Z}_\varphi} \varphi(\epsilon).$$

From Lemma 5.2 and the above estimates

$$\Theta_{\mathbf{S}_\Gamma u}(\epsilon) \leq c\|u\|_{\mathcal{Z}_\varphi} \varphi(\epsilon),$$

similarly

$$\omega_{\mathbf{S}_\Gamma u}(\epsilon) \leq c\|u\|_{\mathcal{Z}_\varphi} \varphi(\epsilon).$$

Furthermore

$$|\mathbf{S}_\Gamma u(z)| \leq 2\Theta_u(d) + \|u\|_{\mathcal{Z}_\varphi} \leq c\|u\|_{\mathcal{Z}_\varphi}, \ z \in \Gamma,$$

$$\|\mathbf{S}_\Gamma u\|_{\mathcal{Z}_\varphi} \leq c\|u\|_{\mathcal{Z}_\varphi},$$

that is, $\mathbf{S}_\Gamma$ is a bounded operator on $\mathcal{Z}_\varphi(\Gamma, \mathbb{R}_{0,m})$. The proof is complete. ∎

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